8th Lecture

CROSS SECTIONS & AMPLITUDES

this lecture will be mostly quantum mechanical

part one: kinematics & quantum dynamics of scattering

part two: introduction to Feynman's path integrals

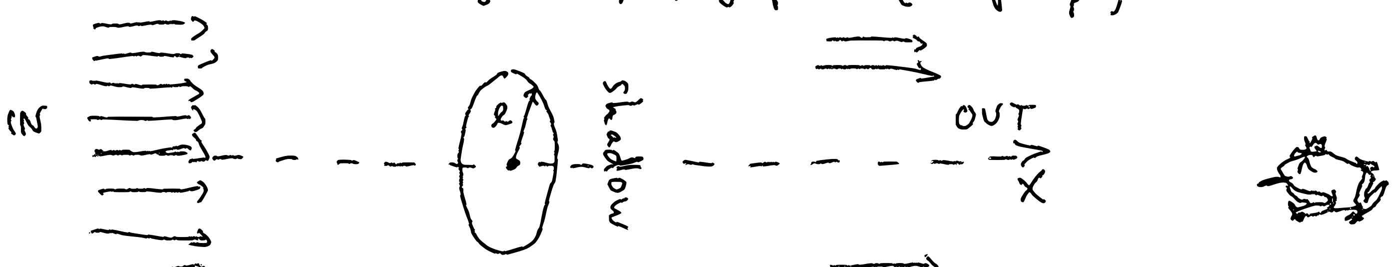
kinematics of scattering

- a flow of mosquitos flying in $+x$ direction

Flux j = # mosquitos through unit yz-area per unit time

a frog at origin catches all mosquitos passing closer than ℓ

assume: $\ell \gg$ frog size, frog fixed (no jumps)



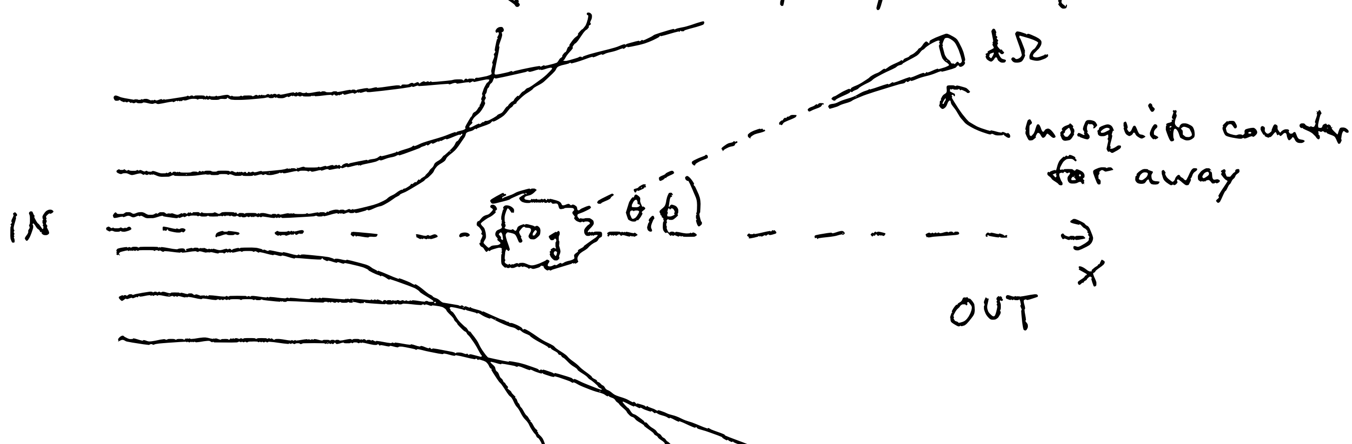
eaten mosquitos/sec $= n = j \cdot \pi l^2 =: j \sigma$

cross section $\sigma = \pi l^2$ inelastic scattering

- modification: frog passive (toy), mosquitos intelligent

$\longrightarrow$ they try to avoid the frog by changing direction

$\longrightarrow$ elastic scattering: no absorption, # mosquitos conserved



how many mosquitos per unit time pass into $d\Omega(\theta, \phi)$?

must be proportional to j & $d\Omega \rightsquigarrow$

$n(d\Omega) = j \frac{d\sigma}{d\Omega} d\Omega$ dim: $\frac{1}{T} = \frac{1}{L^2 \cdot T} \cdot L^2 \cdot 1$

differential cross section $\frac{d\sigma}{d\Omega}$ [dim $= L^2$]

integrate; $\quad n = j \cdot \sigma \qquad \sigma = \int d\Omega \, \frac{d\sigma}{d\Omega}$ total cross section

Rutherford used classical theory & got correct quantum result!

- quantum scattering

probability $\sim |\psi|^2$ $\Longrightarrow$ cross section $\sim |f|^2$

probability amplitude "scattering amplitude"

task: compute probability amplitude for the particle to come from spatial infinity, approach scattering center & go off to infinity but along a different direction

$\rightsquigarrow$ solve time-dependent Schrödinger equation for the evolution of a wave packet with boundary conditions at $t = \pm\infty$

- best done using Feynman's path integral
- traditional method: via solutions of the stationary Schr. equ.

- set-up

$$(\Delta + k^2)\psi = \frac{2mV}{\hbar^2}\psi \; , \quad k^2 = \frac{2mE}{\hbar^2}$$ stationary Schr. eq. in potential $V(\vec{r})$

seek perturbative solution for distances $r \gg a$ = range of V

ansatz: $$\psi(\vec{r}) = e^{ikx} + \frac{1}{r} e^{ikr} f(k,\theta,\phi)$$

check: $(\Delta + k^2)\, e^{ikx} = 0$

but $$\left(\frac{1}{r}\partial_r^2 r + k^2 + \frac{1}{r^2}\Delta_{\theta,\phi}\right)\frac{1}{r} e^{ikr} f = \overbrace{\frac{1}{r^3}}^{\text{fast fall-off}} e^{ikr}\,\Delta_{\theta,\phi} f$$

interpretation: e^{ikx} ~ stationary flow of incoming particles

outgoing spherical wave $\frac{1}{r} e^{ikr} f$ ~ stationary flow of scattered particles

not quite correct: $\nexists$ interference between $e^{ikx} \sim \psi_{in}$ $(t \to -\infty)$

and $\frac{1}{r} e^{ikr} f \sim \psi_{out}$ $(t \to +\infty)$

⇝ should collimate incoming plane wave to a beam by multiplying with a profile function, e.g. $\theta(R^2 - y^2 - z^2)$ (anyway done in experiment)

beam diameter R s.t. $a, \lambda = 2\pi/k \ll R \ll r$ (no diffraction, asymptotics)

⇝ detector away from $\theta = 0$ or π sees only scattered particles ✓

proper normalization $\psi_{in} = \frac{1}{\sqrt{V}} e^{ikx}$

$\psi_{out} = \frac{1}{\sqrt{V}} \frac{1}{r} e^{ikr} f$

(dim. of f is length)

$\rightsquigarrow \vec{j}_{in} = \frac{i\hbar}{2m} \psi_{in} \overleftrightarrow{\nabla} \psi_{in}^* = \frac{1}{V} \frac{\hbar k}{m} \vec{e}_x = \vec{v}/V$

$\vec{j}_{out} = \quad \dots \quad = \frac{1}{V} \frac{\hbar k}{m} \frac{|f|^2}{r^2} \vec{e}_r$

particles passing per unit time into $d\Omega$ at distance r =

$n(d\Omega) = j_{out} r^2 d\Omega = j_{in} \frac{d\sigma}{d\Omega} d\Omega \rightsquigarrow \frac{d\sigma}{d\Omega} = |f(k,\theta,\phi)|^2$

if axisymmetric $\rightsquigarrow$ only θ dependence

Born approximation & beyond

- solve $(\Delta + k^2)\psi = \frac{2mV}{\hbar^2}\psi$ perturbatively in powers of V

as a sequence of corrections $\psi = \psi^{(0)} + \psi^{(1)} + \psi^{(2)} + \dots$ to $\psi^{(0)} \sim e^{ikx}$

$\rightsquigarrow$ infinite chain: $(\Delta + k^2)\psi^{(1)} = \frac{2mV}{\hbar^2}\psi^{(0)}$

$(\Delta + k^2)\psi^{(2)} = \frac{2mV}{\hbar^2}\psi^{(1)}$

$\dots$

at distances $r \gg a$ compare $\psi^{(1)} + \psi^{(2)} + \dots$ with $\frac{1}{r} e^{ikr} f \nearrow \frac{d\sigma}{d\Omega}$

• step 1 Helmholtz eq. $(\Delta+k^2)\psi^{(1)} = \frac{2mV}{\hbar} e^{ikx} =: h(\vec{r})$

first find fundamental inhomogeneous solution = Green's function

$(\Delta + k^2)\, G(\vec{r}) = \underset{\substack{\uparrow \\ \text{convention}}}{-}\, \delta(\vec{r}) \rightsquigarrow G(\vec{r}) = \frac{e^{\pm ikr}}{4\pi r}$ choose + sign for expanding wave

Second realize that homogeneous solution part already fixed by $\psi^{(0)}$

$G(\vec{r})$ is singular at $\vec{r}=0$, nearby $e^{ikr} \approx 1 \to G(\vec{r}) \xrightarrow{r\to 0} \frac{1}{4\pi r}$ Coulomb

$k \to i\mu$ gives static solution to $(\Box+\mu^2)\phi = \delta(\vec{r}) \rightsquigarrow \phi(\vec{r}) \sim \frac{e^{-\mu r}}{4\pi r}$

third, convolute with source function $h(\vec{r})$:

$$\psi^{(1)}(\vec{r}) = -\int d^3r'\, G(\vec{r}-\vec{r}')\, h(\vec{r}') = -\frac{m}{2\pi\hbar^2}\int d^3r' \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(\vec{r}')\, \underset{\psi^{(0)}}{e^{ikx'}}$$

• step 2 simplify for $r \gg a$ hence $|\vec{r}| \gg |\vec{r}'|$

$\frac{1}{|\vec{r}-\vec{r}'|} \to \frac{1}{r}$ & $e^{ik|\vec{r}-\vec{r}'|} \to e^{ikr}\, e^{-ik\vec{e}_r\cdot\vec{r}'}$

$\left[\sqrt{(\vec{r}-\vec{r}')^2} = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + \ldots} \approx r\sqrt{1-2\vec{r}\cdot\vec{r}'/r^2 + \ldots} \approx r\left(1 - \vec{r}\cdot\vec{r}'/r^2 + \ldots\right) \right.$

$$\rightsquigarrow \psi^{(1)}(\vec{r}) = -\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik(\vec{e}_r-\vec{e}_x)\cdot\vec{r}'}\, V(\vec{r}')$$

$$= -\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int d^3r'\, e^{-i\vec{q}\cdot\vec{r}'}\, V(\vec{r}') \quad \text{with } \vec{q} = k(\vec{e}_r - \vec{e}_x) = \vec{k}_{out} - \vec{k}_{in} \text{ momentum transfer}$$

compare with $\frac{1}{r} e^{ikr} f(k,\theta,\phi)$

$$\rightsquigarrow f^{(1)}(k,\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int d^3r' \, e^{-i\vec{q}\cdot\vec{r}'} V(\vec{r}') = -\frac{m}{2\pi\hbar^2}\tilde{V}(\vec{q}) \quad (*)$$

(here depends only on $\vec{q}$)

for a central potential $V(\vec{r}) = V(r)$: only $|\vec{q}| = 2k\sin\theta/2$

result $(*)$ is called the "Born approximation"

- recall Yukawa potential $V(\vec{r}) = -\frac{g^2}{4\pi r} e^{-\mu r} \rightsquigarrow f(\vec{q}) = \frac{mg^2}{2\pi(\vec{q}^2+\mu^2)}$

- step 3 iterate to find $\psi^{(2)}$

$$\psi^{(2)}(\vec{r}) = -\frac{m}{2\pi\hbar^2}\int d^3r' \, \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(\vec{r}')\, \psi^{(1)}(\vec{r}')$$

$$= \left(\frac{m}{2\pi\hbar^2}\right)^2 \int d^3r' \int d^3r'' \, \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(\vec{r}') \, \frac{e^{ik|\vec{r}'-\vec{r}''|}}{|\vec{r}'-\vec{r}''|} V(\vec{r}'')\, e^{ikx''}$$

make same simplifications for $r \gg a$ and represent

$$V(\vec{r}') = \frac{1}{(2\pi)^3}\int d^3q' \, \tilde{V}(\vec{q}')\, e^{i\vec{q}'\cdot\vec{r}'} \quad , \quad V(\vec{r}'') = \frac{1}{(2\pi)^3}\int d^3q'' \, \tilde{V}(\vec{q}'')\, e^{i\vec{q}''\cdot\vec{r}''}$$

inserting all this gives

$$f^{(2)} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \frac{1}{(2\pi)^6} \int d^3r' \int d^3r'' \int d^3q' \int d^3q'' \, \tilde{V}(\vec{q}\,')\tilde{V}(\vec{q}\,'') \, e^{i\vec{q}\,'\cdot\vec{r}\,' + i\vec{q}\,''\cdot\vec{r}\,''} \, e^{-ik\vec{e}_r\cdot\vec{r}\,' + ik\vec{e}_x\cdot\vec{r}\,''} \frac{e^{ik|\vec{r}\,'-\vec{r}\,''|}}{|\vec{r}\,'-\vec{r}\,''|}$$

trick: $$\frac{e^{ik|\vec{r}\,'-\vec{r}\,''|}}{|\vec{r}\,'-\vec{r}\,''|} = \frac{1}{2\pi^2}\int d^3s \, \frac{e^{i\vec{s}\cdot(\vec{r}\,'-\vec{r}\,'')}}{\vec{s}^{\,2}-k^2}$$ Fourier trsfm. of Yukawa for $\mu = -ik$

Small cheat: pole at $s^2 = k^2$, must shift it to define the integral unambiguously: (outgoing wave)

$$\lim_{\varepsilon \to +0} \frac{1}{2\pi^2}\int d^3s \, \frac{e^{i\vec{s}\cdot(\vec{r}\,'-\vec{r}\,'')}}{\vec{s}^{\,2}-(k+i\varepsilon)^2} =: \frac{1}{2\pi^2}\int d^3s \, \frac{e^{i\vec{s}\cdot(\vec{r}\,'-\vec{r}\,'')}}{\vec{s}^{\,2}-k^2-i0}$$

can perform integrals over $d^3r' \, d^3r''$ & d^3s and get

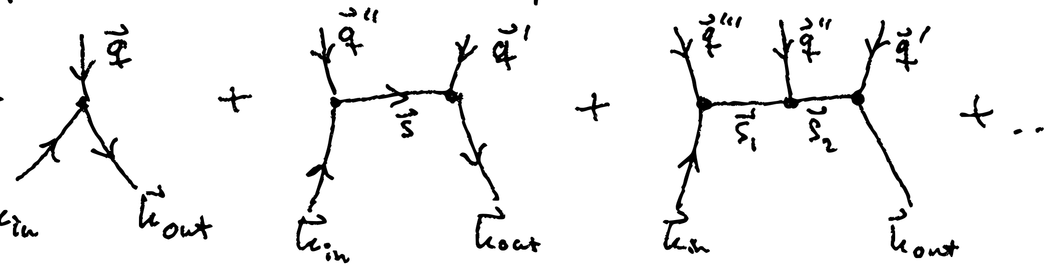
$$f^{(2)} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \int d^3q' \int d^3q'' \, \delta(\vec{q}\,'+\vec{q}\,''-\vec{q}) \, \frac{\tilde{V}(\vec{q}\,')\,\tilde{V}(\vec{q}\,'')}{2\pi^2\left[(\vec{k}_{in}+\vec{q}\,'')^2 - \vec{k}_{out}^{\,2} - i0\right]}$$

depends on $\vec{q}$ and $\vec{k}_{in}$ | $\vec{q} = \vec{k}_{out} - \vec{k}_{in} = k(\vec{e}_r - \vec{e}_x)$ $\quad \vec{k}_{in}^{\,2} = \vec{k}_{out}^{\,2} = k^2$

- Step 4 conditions for valid approximation, $|\psi^{(2)}| < |\psi^{(1)}|$

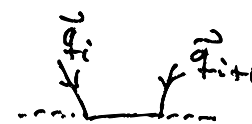
$$\tilde{V} \ll \begin{cases} \dfrac{\hbar^2 k}{ma} & \text{if } ka \gg 1 \\ \dfrac{\hbar^2}{ma^2} & \text{otherwise} \end{cases}$$

· step 5 graphical representation of the series

$f =$ 

with rules:

- blob $= \tilde{V}(\vec{q}_i)$
- beyond first graph: $\int \prod_i \frac{d^3 q_i}{(2\pi)^3} \cdot (2\pi)^3\, \delta\Big(\sum_i \vec{q}_i - \underbrace{\vec{q}}_{\vec{k}_{out}-\vec{k}_{in}}\Big)$ ....
  (no integral in 1st graph)
- momenta/wave vectors are conserved at each vertex,
  e.g. 2nd graph: $0 = \vec{k}_{in} + \vec{q}\,'' - \vec{s} = \vec{s} + \vec{q}\,' - \vec{k}_{out}$
- horizontal line = nonrelativistic propagator



$\hbar \vec{s}_i = \vec{p}_i$

$$G(\vec{p}_i) = \frac{2m}{\hbar^2} \frac{1}{k^2 - \vec{s}_i^{\,2} + i0} = \frac{1}{E - \frac{\vec{p}_i^{\,2}}{2m} + i0}$$

$E = \hbar^2 k^2 / 2m$

- each graph is multiplied by overall factor $-\frac{m}{2\pi\hbar^2}$

$\rightsquigarrow$ nonrelatistic or "old-fashioned" diagram technique

- Some remarks

– characteristic factor $\sim \frac{1}{E_n - E_m}$ is energy denominator from stationary perturbation theory for $H = H_0 + V$

– heuristic picture of 2^{nd} order correction:

$$\begin{array}{ccccc} |in\rangle & \xrightarrow[\langle n|V|in\rangle]{} & |n\rangle & \xrightarrow[\langle out|V|n\rangle]{} & |out\rangle \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\ \hbar \vec{k}_{in} & \tilde{V}(\vec{q}\,'') & \vec{p} = \hbar \vec{s} & & \hbar \vec{k}_{out} \end{array}$$

system lives in an intermediate state $|n\rangle$ for time $\frac{\hbar}{\Delta E}$

sum over all intermediate state $\rightarrow$ integrate over intermediate momenta

– nonrelativistic propagator $G(\vec{p}_i)$ is the Fourier transform of the fundamental solution $G(\vec{r}) = \frac{e^{ikr}}{4\pi r}$ of Helmholtz eq.

But Helmholtz eq. = Schrödinger eq. (with V on RHS) in momentum space: $(E - \vec{p}^{\,2}/2m)\, \tilde{\psi}(p) = \text{source}$

$\rightsquigarrow$ propagator $G(\vec{p})$ is just the inverse $(E - \vec{p}^{\,2}/2m)^{-1}$

– relativistic rewriting $\longrightarrow$ Feynman perturbation theory

will not derive this rigorously but announce its rules and argue heuristically by nonrelativistic analogies

Path integrals

- nonrelativistic approach inconvenient QFT

two methods < operator approach (Schwinger) / path-integral approach (Feynman)

nowadays, especially for more complicated QFTs like YM, the path-integral method is superior

here: will only illustrate for quantum mechanics

- consider quantum mechanics with single d.o.f. (for simplicity)

solve time-dependent Schr. eq.: $i\hbar \partial_t \psi(q,t) = \hat{H}\,\psi(q,t)$

with $\hat{H}(\hat{p},q)$ and $\partial_t H = 0$ & initial condition $\psi(q,t_0) = \psi_0(q)$

$$\psi(q,t) = \hat{U}(t-t_0)\,\psi(q,t_0) \quad \text{with} \quad \hat{U}(t-t_0) = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}}$$

"time evolution operator"

consider its kernel = matrix element (for $t_1 - t_0 = \Delta t \geq 0$)

$$K(q^{(1)}, q^{(0)}; t_1 - t_0) = \langle q^{(1)} |\, \hat{U}(t_1 - t_0)\, | q^{(0)} \rangle \equiv \langle q^{(1)}, t_1 | q^{(0)}, t_0 \rangle$$

- K describes the probability amplitude that system is found at $q = q^{(1)}$ for $t = t_1$, provided it was located at $q = q^{(0)}$ for $t = t_0$ earlier. Rewrite:

$$\langle q^{(1)} | \psi(t_1) \rangle = \langle q^{(1)} | \hat{U}(t_1 - t_0) | \psi(t_0) \rangle = \int_{-\infty}^{+\infty} dq^{(0)} \langle q^{(1)} | \hat{U}(t_1 - t_0) | q^{(0)} \rangle \langle q^{(0)} | \psi(t_0) \rangle$$

$$\parallel$$

$$\psi(q^{(1)}, t_1) = \int dq^{(0)} \, K(q^{(1)}, q^{(0)}; t_1 - t_0) \, \psi(q^{(0)}, t_0)$$

spectral decomposition:

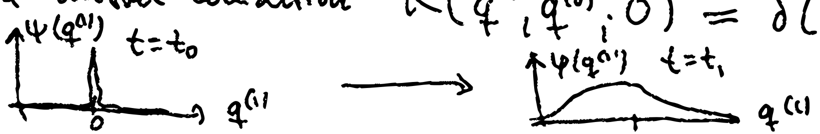
$$K(q^{(1)}, q^{(0)}; \Delta t) = \sum_k \psi_k(q^{(1)}) \psi_k^*(q^{(0)}) \, e^{-\frac{i}{\hbar} E_k \Delta t}$$

where $\hat{H} \psi_k(q) = E_k \psi_k(q)$.

- K is the fundamental homogeneous solution for the time dep't Schrödinger equation:

$$\left(i\hbar \partial_t - \hat{H}(\hat{p}^{(1)}, q^{(1)}) \right) K(q^{(1)}, q^{(0)}; \Delta t) = 0$$

with initial condition $K(q^{(1)}, q^{(0)}; 0) = \delta(q^{(1)} - q^{(0)})$



for $\psi(q^{(0)}, t) = \delta(q^{(0)})$

- explicit solution for free particle (of mass m, in $D=3$):

$$K(\vec{q}^{(1)}, \vec{q}^{(0)}; \Delta t) = \left(\frac{m}{2\pi\hbar\Delta t}\right)^{3/2} e^{\frac{im(\vec{q}^{(1)}-\vec{q}^{(0)})^2}{2\hbar\Delta t}}$$

[analogous to solution of diffusion/heat equation]
by $t = i\tau$

also solvable analytically for harmonic oscillator

what to do in general? for $\hat{H} = \hat{p}^2/2m + V(\hat{q})$?

- Feynman's insight: K is also given by a path integral

$$K(q^{(1)}, q^{(0)}; \Delta t) = \int \left(\prod_t dq(t)\right) e^{\frac{i}{\hbar}\int_{t_0}^{t_1} dt\, L(q, \dot{q})} \quad \text{with} \begin{cases} q(t_0) = q^{(0)} \\ q(t_1) = q^{(1)} \end{cases}$$

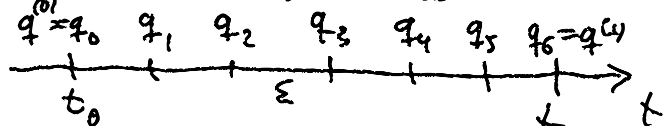
note: integration over all coordinate values at each intermediate time t!

continuously many integrations, controllable as limit of finitely many:

$\Delta t = n \cdot \varepsilon$, $n \to \infty$, $\varepsilon \to 0$ & $\int \prod_t dq(t) = \lim_{n\to\infty} \underbrace{C_n}_{\text{some constants}} \int \prod_{k=1}^{n-1} dq_k \quad \begin{cases} q_k = q(t_0 + k\varepsilon) \\ 1 \le k \le n-1 \end{cases}$

$$\int_{t_0}^{t} dt\, L(q, \dot{q}) \to \sum_{k=1}^{n} \varepsilon L\left(q_{k-1}, \frac{q_k - q_{k-1}}{\varepsilon}\right)$$

difference approx. to $\dot{q}(t_0 + (k-1)\varepsilon)$



example $n = 6$

• example $\quad L = \frac{1}{2}\dot{q}^2 - V(q)$

$$K(q^{(1)}, q^{(0)}; \Delta t) = \lim_{n\to\infty} C_n \int \prod_{k=1}^{n-1} dq_k \ \ldots$$

$$\times \exp\left\{\frac{i}{\hbar}\left[\frac{(q^{(1)}-q_{n-1})^2}{2\varepsilon} + \ldots + \frac{(q_1-q^{(0)})^2}{2\varepsilon} - \varepsilon V(q_{n-1}) - \ldots - \varepsilon V(q^{(0)})\right]\right\}$$

• heuristic derivation for this example

set $t_0 = 0$, $t_1 = \Delta t = t$

$$K = \langle q^{(1)}, t \,|\, q^{(0)}, 0\rangle = \int_{-\infty}^{+\infty} dq_* \langle q^{(1)}, t \,|\, q_*, \tfrac{t}{2}\rangle\langle q_*, \tfrac{t}{2} \,|\, q^{(0)}, 0\rangle$$

by insertion of resolution of unity

iterate by subdividing time interval as $t = n\cdot\varepsilon \Rightarrow$

$$K = \prod_{k=1}^{n-1}\int_{-\infty}^{+\infty} dq_k \, \langle q^{(1)}, t \,|\, q_{n-1}, t-\varepsilon\rangle\langle q_{n-1}, t-\varepsilon \,|\, q_{n-2}, t-2\varepsilon\rangle \ldots \langle q_1, \varepsilon \,|\, q^{(0)}, 0\rangle$$

for small ε : $\langle q_1, \varepsilon \,|\, q^{(0)}, 0\rangle \approx \sqrt{\frac{m}{2\pi\hbar\varepsilon}}\, e^{\frac{im(q_1-q^{(0)})^2}{2\hbar\varepsilon}}\, e^{-\frac{i}{\hbar}\varepsilon V(q^{(0)})}$

multiply all these factors & take limit $n\to\infty$, $\varepsilon \to 0$

and adjusting C_n to cancel infinity from $\left(\frac{m}{2\pi\hbar\varepsilon}\right)^n$ gives result ✓

- opto-mechanical analogy

optics	mechanics
wave packet	particle
ray	classical trajectory
geometric optics	classical mechanics
wave optics	quantum mechanics
refraction index	potential energy
Fermat's principle	Maupertuis' principle
Huygens-Fresnel principle	quantum transition amplitude
$\downarrow$	$\downarrow$
electromagnetic amplitude emitted at A & detected at B = sum over all possible paths of rays between A and B, weighted by phase $e^{2\pi i l/\lambda}$	path integral = Sum over all possible paths of trajectories between $q^{(0)}$ & $q^{(1)}$, weighted by phase $e^{\frac{i}{\hbar}S}$
[l = optical path length, λ = wavelength]	[$S = \int_0^1 dt\, L$ = action of the path]

- semiclassical limit: $\hbar \to 0$ $\rightsquigarrow$ phases large $\rightsquigarrow$ wild oscillation! destructive interference of nearby paths, except when $\delta S = 0$ $\rightsquigarrow$ classical trajectories dominate the path integral | mathematics: $t = -i\tau$ Wiener integral ✓